Recall $\mathcal{M}_n : S/\mathbb{Z}[\frac{1}{n}] \longmapsto \{(E,\alpha)\}/\cong$

α level-n-str.

$\widetilde{\mathcal{M}}_n : S/\mathbb{Z}[\frac{1}{6n}] \longmapsto \{(E,\alpha,\pi)\}/\cong$

Thm $n \geq 3$. Then $\mathcal{M}_n$ representable by aff. scheme.

We today prove this for $\mathbb{Z}[\frac{1}{6n}] \otimes_{\mathbb{Z}[\frac{1}{n}]} \mathcal{M}_n$.

Consider

$$\begin{tikzcd} & \widetilde{\mathcal{M}}_n \arrow[dl] \arrow[dr] & \\ \mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n & & \mathcal{M}_n \end{tikzcd}$$

Last time: $\mathcal{M}_n \cong \mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n$ if we can show that

$\mathbb{G}_m \curvearrowright \widetilde{\mathcal{M}}_n$ freely.

Valuative criterion for $\mathbb{G}_m \times \widetilde{\mathcal{M}}_n \longrightarrow \widetilde{\mathcal{M}}_n \times \widetilde{\mathcal{M}}_n$:

To show R DVR, $K = \operatorname{Frac} R$.

$(E,\alpha,\pi), (E',\alpha',\pi') \in \widetilde{\mathcal{M}}_n(R)$

s.th. $\exists$ iso $\phi_K : (E,\alpha)_K \xrightarrow{\cong} (E',\alpha')_K$ with
$\phi^*(\pi') = \lambda \cdot \pi \quad \lambda \in K^\times$

Then $\exists! \ \phi : (E,\alpha) \xrightarrow{\cong} (E',\alpha')$ lifting ϕ_K & $\lambda \in R^\times$.

Assume we have $\phi : E \xrightarrow{\cong} E'$ lifting ϕ_K.

Then $\cdot)$ ϕ is unique

$\cdot)$ $\phi \circ \alpha = \alpha'$

both hold since $E', E'[n] / R$ are separated

& $E_K, (\underline{\mathbb{Z}/n}_R)_K$ schematically dense.

Furthermore, $\phi^* \pi' = \mu \cdot \pi$ for some $\mu \in R^\times$.

Since $\Gamma(E, \Omega') \subset \Gamma(E_K, \Omega'_K)$,

$\mu = \lambda \in R^\times$ as claimed.

So everything boils down to:

<u>Thm</u> (Weil) S Dedekind, connected, η gen pt.

Then $\mathrm{Hom}(E, E') \xrightarrow{\cong} \mathrm{Hom}(E_\eta, E'_\eta)$.

<u>Thm (Stronger variant, BLR 4.4 Thm 1)</u>

S normal, noetherian, $u : Z \dashrightarrow G$ S-rational

map from smooth S-scheme Z to separated grp sch. G.

Then (u defined in codim 1 $\Longrightarrow$ defined everywhere).

Proof of Weil Extension Thm

Idea Given $H, G/_k$ group schemes + $\phi : U \to G$ defined on open dense $U \subseteq H$ that satisfies group homomorph identity on $m_H^{-1}(U) \cap (U \times U)$.

Then use translation + patching to extend to $H \to G$.

Important principles

1) X/S separated, $U \subset Y/S$ schem. dense.

Then $\mathrm{Hom}_S(Y, X) \hookrightarrow \mathrm{Hom}_S(U, X)$.

2) X loc fin pres. $y \in Y$. Then [Stacks 01ZC]

$$\varinjlim_{\substack{y \in U \\ \text{open}}} \mathrm{Hom}_S(U, X) \xrightarrow{\cong} \mathrm{Hom}_S(\mathrm{Spec}\, \mathcal{O}_{Y,y}, X).$$

Variant $s \in S$. X loc fin pres, $Y \to S$ qcqs

$$\varinjlim_{\substack{s \in U \\ \text{open}}} \mathrm{Hom}_S(U \times_S Y, X) \xrightarrow{\cong} \mathrm{Hom}_S(\mathrm{Spec}\, \mathcal{O}_{S,s} \times_S Y, X).$$

<u>Consequences</u> $\phi_\eta : E_\eta \to E'_\eta$ given

·) If ϕ lifts, then uniquely

$\Rightarrow$ If ϕ lifts to $U_i \xrightarrow{\phi_i} E'$

$E = \cup U_i$ open covering, then ϕ_i glue to lift ϕ.

·) ϕ lifts $\Leftrightarrow$ $\forall s \in S$, $\exists$ lift

$$\phi_s \quad \operatorname{Spec} \mathcal{O}_{S,s} \underset{S}{\times} E \longrightarrow \operatorname{Spec} \mathcal{O}_{S,s} \underset{S}{\times} E'.$$

$\Rightarrow$ wlog $S = \operatorname{Spec} R$, R DVR.

<u>Heart of Proof</u>:

$S = \operatorname{Spec} R = \{s, \eta\}$

1) $x^* \in E_s$ generic point. Then $\mathcal{O}_{E,x^*}$ is DVR since

cod 1 point in regular scheme.

Valuative Criterion $\Rightarrow$ $\exists$ extension $\operatorname{Spec} \mathcal{O}_{E,x^*} \longrightarrow E'$

Principle 2) $\Rightarrow$ $\exists$ open $U \subseteq E$ s.th. $\underset{=2}{\operatorname{codim}_E(E \setminus U)}$

+ extension $\psi : U \longrightarrow E'$

2) Assume that $E \setminus U = \{\overline{a}_1, \dots, \overline{a}_r\} \subseteq E(\kappa(s))$

consists of rational points.

Assume further $\exists \{a_1, \dots, a_r\} \in E(R)$ s.th.

$\overline{a_i} = a_i(s)$.

Assume $\exists\ b \in E(R)$ s.th. $b + a_i \in U(R)\ \forall i$.

Then $t_{\varphi(b)}^{-1} \circ \varphi \circ t_b : t_b^{-1}(U) \longrightarrow E'$

agrees with φ on $t_b^{-1}(U) \cap U$ since φ_η is

is a group homomorphism.

$t_b^{-1}(U) \cup U = E$, so they glue to $E \xrightarrow{\phi} E'$.

Left to reduce to this situation:

3) <u>Claim</u> If $R \longrightarrow R'$ is faithfully flat ext of DVRs

s.th. $\exists\ \phi' : R' \otimes_R E \longrightarrow R' \otimes_R E'$

lifting (base change of) ϕ_η. Then ϕ' stems from

some ϕ already.

Proof fpqc descent for maps of schemes:

$$\mathrm{Hom}(X,Y) = \mathrm{Eq}\left(\mathrm{Hom}(X',Y) \overset{-\circ p_2}{\underset{-\circ p_1}{\rightrightarrows}} \mathrm{Hom}(X'\times_X X', Y)\right)$$

for any fpqc $X' \to X$.

Apply with $X = E$, $X' = R'\otimes_R E$, $Y = E'$.

Hence to check: $\phi' \circ p_1 = \phi' \circ p_2$.

But $(\phi' \circ p_1)_\eta = \phi_\eta = (\phi' \circ p_2)_\eta$

+ fact that $(R' \otimes_R R' \otimes_R E)_\eta$ schem dense

imply this. $\square$ 3)

4) Claim $\exists$ DVR R' + faithfully flat $R' \to R$

s.th. all assumptions from 2) hold.

Proof $a \in E_s$ any closed point. Then $\kappa(a)/\kappa(s)$

is finite extension, say $\cong \kappa(s)[t]/f(t)$, f monic.

Put R_1 := normalization of an irred comp

of $R[t]/\tilde{f}(t)$ $\tilde{f}$ monic lift.

Then a is image of R_1-point, $R \to R_1$

faithfully flat. Iterating this, find $R \to R_1$ s.th. $(R_1 \otimes_R E) \setminus (R_1 \otimes_R U)$ consists of rational points.

<u>Next</u> $\bar{b} \in E_s(\overline{\kappa(s)}) \setminus \{\bar{a_i} - \bar{a_j}\}_{i,j}$ any.

Above process gives $R_1 \to R_2$ s.th. also $\bar{b}$ rational.

Put $R_3 := \widehat{R_2}$, adic completion.

R_2 DVR $\longrightarrow$ R_3 DVR, complete

& $R_2 \longrightarrow R_3$ faith. flat.

Let $\pi \in R_3$ uniformizer. Given any ring A,

$$\mathrm{Hom}(A, R_3) \xrightarrow{\cong} \varprojlim_i \mathrm{Hom}(A, R_3/\pi^i)$$

by completeness.

Lifting criterion for smoothness of E/R

$\Rightarrow$ $E(R_3/\pi^{i+1}) \twoheadrightarrow E(R_3/\pi^i)$ $\forall i$.

$\Rightarrow$ $\bar{a_i}, \bar{b}$ lift to R_3-points a_i, b. Wohoo!!!!

Then 2) & 3) apply. $\blacksquare$

<u>Fun observation</u> The ring $R_3 \otimes_R R_3$ that occurs during descent is super-complicated!

<u>E.g.</u> $\left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)[p^{-1}] = \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_p$

Contains an idempotent from $K \otimes_{\mathbb{Q}} K$
for <u>every</u> quadratic $K \hookrightarrow \mathbb{Q}_p$
(i.e. p split in K)

It is moreover ∞-dimensional, non-noetherian.

Namely $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is integral over some ring

$$\mathbb{Q}(T_i\ ;\ i\in I) \otimes_{\mathbb{Q}} \mathbb{Q}(T_i\ ;\ i \in I) \qquad \left(\begin{array}{l}\text{choose} \\ \text{transcendence base}\end{array}\right)$$

and these are non-noeth. + ∞-dim.

Cor S Dedekind, connected, $X \to S$ EC or AV.

Then X is a Néron model of X_η.

In other words, for every smooth $T \to S$,

$$\mathrm{Hom}_S(T, X) \xrightarrow{\cong} \mathrm{Hom}_{S_\eta}(T_\eta, X_\eta).$$